Differential geometry I   Week 3

Last time: We saw the definition of the length of a (piecewise) C^1 curve $\gamma: I \to \mathbb{R}^n$.

$$\ell(\gamma) = \int_I |\dot{\gamma}| \, dt.$$

Proposition:

If $\gamma : [a,b] \to \mathbb{R}^n$ is C^1, then $d(\gamma(a), \gamma(b)) \leq \ell(\gamma)$

(So the straight line segment has the minimum length among curves connecting two points)

Proof:

For each coordinate $j = 1, \dots n$, we have $\gamma_j(a) - \gamma_j(b) = \int_a^b \frac{d\gamma_j(t)}{dt} dt$

So $d(\gamma(a), \gamma(b)) = \|\gamma(a) - \gamma(b)\| = \left\| \int_a^b \frac{d\gamma(t)}{dt} dt \right\|$ ①

From the triangle inequality $\left\| \sum_{k=0}^{N} x_k \right\| \leq \sum_{k=0}^{N} \|x_k\|$

we get (in the limit of a partition): $\left\| \int_a^b x(t) \, dt \right\| \leq \int_a^b \|x(t)\| \, dt$

for any C^0 function $x : [a,b] \to \mathbb{R}^n$

So: $\left\| \int_a^b \frac{d\gamma}{dt} dt \right\| \leq \int_a^b \left\| \frac{d\gamma}{dt} \right\| dt = \ell(\gamma)$

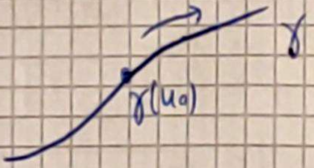
From ①: QED.

There is a natural parametrization of a curve in which many of its geometric properties can be easily read off:

Definition: Let $\gamma : [a,b] \to \mathbb{R}^n$ be of class C^1 and let $u_0 \in [a,b]$

Natural parameter (or arc length parameter or curvilinear abscissa):

$$S_\gamma(u) = \int_{u_0}^{u} v_\gamma(u)\, du = \int_{u_0}^{u} \|\dot{\gamma}(u)\|\, du \qquad (\gamma(u_0): \text{Initial point})$$



parametrize the curve by the length of its arc starting from $\gamma(u_0)$.

$$S_0 \quad S_\gamma(u) = \begin{cases} \ell(\gamma|_{[u_0,u]}), & u \geq u_0, \\ -\ell(\gamma|_{[u,u_0]}), & u \leq u_0 \end{cases}$$

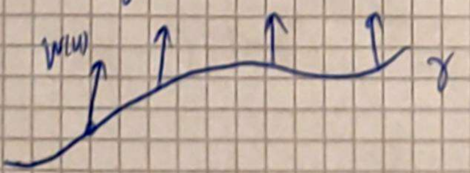
- As we will see, we will often parametrize a curve γ by S_γ - this will be called the natural or unit speed parametrization

Vector fields along a curve:

Definition: A vector field along a curve $\gamma : I \to \mathbb{R}^n$ is a vector-valued map

$$W(u) = \sum_{i=1}^{n} \underset{\uparrow\ \text{function}}{w_i(u)} \cdot e_i \quad , u \in I.$$

We thing of $W(u)$ as having base point $\gamma(u)$.



- W is of class C^k if the W_i's are C^k functions

- Space of vector fields along γ: Γ_γ (usually we restrict to smooth ones, but depends on the context)

Examples: 1) If γ is C^1: the velocity $u \longrightarrow \dot{\gamma}(u)$

2) If γ is C^2: the acceleration $u \longrightarrow \ddot{\gamma}(u)$

3) If $W, Z \in \Gamma_\gamma$; $f, g : I \longrightarrow \mathbb{R}$:

$$f \cdot W + g \cdot Z \in \Gamma_\gamma$$

(So Γ_γ is a "module" over the ring of function over I)

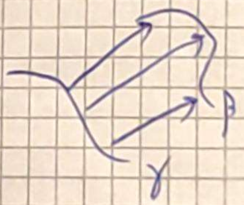
4) If W is of class $C^k \Rightarrow \dot{W}$ is of class C^{k-1}

5) In dimension 3: $u \longrightarrow Z(u) \times W(u)$ is also a vector field.

6) If $\beta, \gamma : I \rightarrow \mathbb{R}^n$ are two curves:

$$W(u) = \beta(u) - \gamma(u)$$

is a vector field along γ

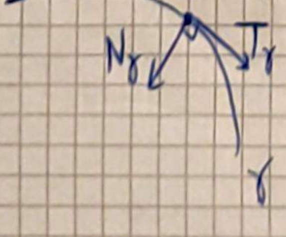


7) If γ is regular:

Unit tangent vector $T_\gamma(u) = \dfrac{\dot{\gamma}(u)}{\|\dot{\gamma}(u)\|}$

8) If γ is biregular: **Principal normal vector**

$$N_\gamma(u) = \frac{\ddot{\gamma}(u) - \langle \ddot{\gamma}(u), T_\gamma(u) \rangle T_\gamma(u)}{\| \ddot{\gamma}(u) - \langle \ddot{\gamma}(u), T_\gamma(u) \rangle T_\gamma(u) \|}$$



Exercise: $\{T_\gamma, N_\gamma\}$ orthonormal, they span the oscullating plane.

Lemma (Leibniz rule):

Let W, Z be vector fields of class C^1 across $\gamma : I \to \mathbb{R}^n$, then

$$\frac{d}{dt}\langle W, Z\rangle = \langle \dot{W}, Z\rangle + \langle W, \dot{Z}\rangle$$

- When $n=3$: $\frac{d}{dt}(W \times Z) = \dot{W} \times Z + W \times \dot{Z}$
- When $n=2$: $\frac{d}{dt}(W \wedge Z) = \dot{W} \wedge Z + W \wedge \dot{Z}$

(and similarly for other bilinear expressions).

Proof: Either directly in coordinates, or use the fact that, for any such bilinear expression (focus on $\langle \cdot, \cdot \rangle$ for simplicity):

$$\frac{d}{dt}\langle W(t), Z(t)\rangle = \lim_{\varepsilon \to 0} \frac{\langle W(t+\varepsilon), Z(t+\varepsilon)\rangle - \langle W(t), Z(t)\rangle}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0}\left(\frac{\langle W(t+\varepsilon), Z(t+\varepsilon) - \langle W(t+\varepsilon), Z(t)\rangle + \langle W(t+\varepsilon), Z(t)\rangle - \langle W(t), Z(t)\rangle}{\varepsilon} \right)$$

$$= \lim_{\varepsilon \to 0}\left(\left\langle W(t+\varepsilon), \frac{Z(t+\varepsilon) - Z(t)}{\varepsilon} \right\rangle + \left\langle \frac{W(t+\varepsilon) - W(t)}{\varepsilon}, Z(t) \right\rangle \right)$$

$$= \langle W(t), \dot{Z}(t)\rangle + \langle \dot{W}(t), Z(t)\rangle. \quad \square$$

Corollary:

- $\langle W_1(t), W_2(t)\rangle = \text{const} \rightarrow \langle \dot{W}_1, W_2\rangle = -\langle W_1, \dot{W}_2\rangle$
- $\|W_1\| = \text{const} \Rightarrow \langle \dot{W}_1, W_1\rangle = 0.$

Change of parametrization for a curve

Defn. Let $\alpha : I \to \mathbb{R}^n$ be a parametrized curve. A direct reparametrization (or reparametrization in the same direction) is a curve $\beta : J \to \mathbb{R}^n$ such that $\exists$ bijection $h : I \to J$ such that:

1) $h \in C^1$
2) $h' > 0$
3) $\alpha = \beta \circ h$

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[dr, "\alpha"'] & J \arrow[d, "\beta"] \\ & \mathbb{R}^n \end{tikzcd}$$

If $h' < 0$: indirect reparametrization or inversion.

Note:

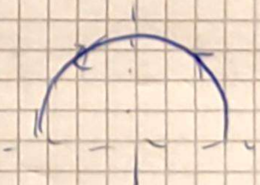
- $\alpha(I) = \beta(J)$ (same trace)
- $\dot{\alpha} = \dfrac{d}{dt}(\beta(h(t))) = h'(t) \cdot \dot{\beta}(h(t))$ so $\dot{\alpha} \,//\, \dot{\beta}$

($\dot{\alpha} \uparrow\uparrow \dot{\beta}$ if $h' > 0$
$\dot{\alpha} \uparrow\downarrow \dot{\beta}$ if $h' < 0$)

- $h' \neq 0$: α, β have the same singular points.

Example : $\alpha : [0, \pi) \to \mathbb{R}^2$, $\alpha(\theta) = (\cos\theta, \sin\theta)$

$\beta : (-1, 1) \to \mathbb{R}^2$, $\beta(x) = (x, \sqrt{1-x^2})$



Both parametrize the half circle.

If $h : (0, \pi) \to (-1, 1)$, $h(\theta) = \cos\theta$

then $\underset{\text{bijection}}{h \in C^1}$, $h' < 0$ (inversion), $\beta \circ h = \alpha$

Note:
- Kinematic properties of a curve: Depending on the parametrization
- Geometric properties: Independent of a parametrization.

Lemma: The $\overset{\text{unit}}{\vee}$ tangent vector T_a of a curve a is geometric (up to a sign)

Proof: Let β be a reparametrization of a, i.e. $a = \beta \circ h$. Assume it's a direct reparametrization ($h' > 0$)

At the point $a(t)$:

$$T_a(t) = \frac{\dot{a}(t)}{\|\dot{a}(t)\|} = \frac{\frac{d}{dt}(\beta(h(t)))}{\|\frac{d}{dt}(\beta(h(t)))\|} = \frac{\cancel{h'(t)} \cdot \dot{\beta}(h(t))}{\cancel{|h'(t)|} \cdot \|\dot{\beta}(h(t))\|}$$

At the point $\beta(h(t)) = a(t)$:

$$T_\beta(h(t)) = \frac{\dot{\beta}(h(t))}{\|\dot{\beta}(h(t))\|} = T_a(t)$$

If $h' < 0$: $T_a(t) = -T_\beta(h(t))$. ∎

Lemma: The length of a curve is a geometric quantity

Proof: Let $a: I \to \mathbb{R}^n$, $\beta: J \to \mathbb{R}$ s.t

$a = \beta \circ h$, $h' > 0$ (same proof if $h' < 0$)

Assume $I = [c, d]$, $J = [c', d']$,

so that (since $h' > 0$, h bijection)

$h(c) = c'$, $h(d) = d'$.

Then:

$$\ell(\alpha) = \int_c^d \|\dot{\alpha}(t)\|\, dt$$

$$= \int_c^d \left\| \frac{d}{dt}(\beta(h(t))) \right\| dt = \int_c^d |h'(t)| \cdot \|\dot{\beta}(h(t))\|\, dt$$

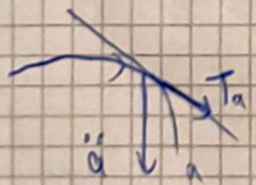
$$\overset{h'(t)>0}{\underset{s=h(t)}{=}} \int_{h(c)}^{h(d)} \|\dot{\beta}(s)\|\, ds = \ell(\beta) \quad \blacksquare$$

(similar proof if $h' < 0$, whence $h(c) = d'$ and $h(d) = c'$)

Note: The acceleration is not geometric:

$$\ddot{\alpha}(t) = \frac{d^2}{dt^2}(\beta(h(t))) = (h'(t))^2 \cdot \ddot{\beta}(h(t)) + h''(t) \cdot \dot{\beta}(t)$$

Note: The change in the acceleration is in a direction // to T_α and a factor $(h')^2 > 0$. So



The fact that $\ddot{\alpha}$ lies on that side of the tangent line is a geometric property.

Given the above properties:
Is there a "preferred", natural parametrisation, that allows us to read off geometric properties more easily?

Natural (or unit speed) parametrization of a regular curve.

Theorem: Let $\alpha : I \to \mathbb{R}^n$ be a regular curve and $t_0 \in I$. There exists a unique direct reparametrization $h : I \to J$ with $0 \in J$ and $h(t_0) = 0$ such that the curve

$\tilde{\alpha} = \alpha \circ h^{-1}$ has unit speed i.e. $V_{\tilde{\alpha}}(s) = 1$ for $s \in J$.

In this case: $s = h(t)$ is the arc length parameter of α.

Proof: Since $\alpha = \tilde{\alpha} \circ h$

$$\Rightarrow \quad \dot{\alpha}(t) = h'(t) \cdot \dot{\tilde{\alpha}}(h(t))$$

$$\left.\begin{matrix} V_{\tilde{\alpha}} = 1 \\ \& \quad h'(t) > 0 \end{matrix}\right\} \|\dot{\alpha}(t)\| = h'(t) \quad , t \in I$$

So: Since $h(t_0) = 0 \quad \Rightarrow \quad h(t) = \int_{t_0}^{t} \|\dot{\alpha}(t)\| \, dt$

$$= S_\alpha(t)$$

(unique process)

Remark: Many of the results we will see later will be stated in terms of the natural parametrization of a curve.

Method for finding the natural parametrization of a given curve $t \longrightarrow \gamma(t)$:

1) Choose initial point t_0, calculate the arc length parameter $S_\gamma(t) = \int_{t_0}^{t} \|\dot{\gamma}(\tau)\| \, d\tau$

2) The change of parameter $t \to S_\gamma(t)$ is the change to the natural parametrization.

Invert the relation to get $t \to s(t)$

Then $\tilde{\alpha}(s) = \alpha(t(s))$.

Example:

$x \in (-1,1)$

For the half circle $\gamma(x) = (x, \sqrt{1-x^2})$: $\dot{\gamma}(x) = \left(1, -\frac{x}{\sqrt{1-x^2}}\right)$

$V_\gamma(x) = \sqrt{1 + \frac{x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$. If $x_0 = 0$:

So $S_\gamma(x) = \int_0^x V_\gamma(\tilde{x}) \, d\tilde{x} = \int_0^x \frac{1}{\sqrt{1-p^2}} \, dp \overset{p=\cos t}{=} \int_{\pi/2}^{\cos^{-1}(x)} \frac{1}{\sin t} \cdot (-\sin t) \, dt$

$= \frac{\pi}{2} - \mathrm{Arccos}(x)$

Invert: $x = \cos\left(\frac{\pi}{2} - s\right)$, $s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So $\tilde{\gamma}(s) = \gamma(x(s)) = \left(\cos\left(\frac{\pi}{2} - s\right), \sqrt{1 - \cos^2\left(\frac{\pi}{2} - s\right)}\right) = (\sin(s), \cos(s))$.

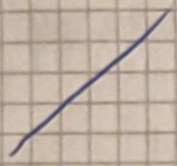
For the change of parameter $t \to s(t)$:

$\frac{ds}{dt} = V_\gamma(t)$ so $ds = V_\gamma(t) \, dt$

$\frac{d}{ds} = \frac{1}{V_\gamma(t)} \frac{d}{dt}$

We have only discussed the case of regular (i.e. C^1) change of parameters. Under such a change: The singular points of a curve remain singular.

However: If $\gamma(t) = (t^3, t^3)$ ← $\dot{\gamma}(0) = (0,0)$ singular



but if I perform the change

$$u = \begin{cases} t^{1/3}, & t \geq 0 \\ -(-t)^{1/3}, & t \leq 0 \end{cases}$$

then $\tilde{\gamma}(u) = \gamma(t(u)) = (u,u)$ is regular at 0.

In general: Switching to the natural parameter s_γ: I can decide if the singularity is "removable" or a true singularity. (If γ admits a parametrization for which it is a regular curve. It is also regular in the natural parameter)

E.g.:

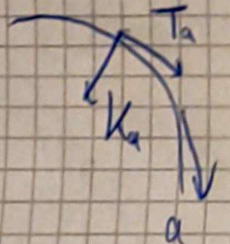
← For the natural parametrization: $\dot{\gamma}$ has a jump discontinuity

Curvature of a curve in $\mathbb{R}^n$:

Def: Let $a: I \to \mathbb{R}^n$ be of class C^2. Curvature vector:

$$K(a,t) = \underbrace{\frac{1}{V_a(t)} \frac{d}{dt}}_{\frac{d}{ds} \text{ if s is the natural paramerer}} T(a,t)$$

← unit tangent vector: $T = \frac{1}{V_a} \cdot \dot{a}$



(Scalar) curvature: $\kappa(a,t) = \|K(a,t)\|$

Note: Since $\|T_a(t)\| = 1 \Rightarrow K_a \perp T_a$.

Remark: The curvature vector and the curvature are geometric quantities (i.e. independent of the parametrization).

(Note: $\frac{1}{V_a}\cdot\frac{d}{dt}$ is a "geometric" derivative, since it is the derivative with respect to the natural parameter)

Fundamental example:

A circle with center at p, radius $R>0$ in the plane spanned by the orthonormal vectors $\{e_1, e_2\}$:

$$\gamma(\theta) = p + R\cos\theta\cdot e_1 + R\cdot\sin\theta\cdot e_2$$

$$\dot{\gamma}(\theta) = -R\cdot\sin\theta\, e_1 + R\cdot\cos\theta\cdot e_2 \Rightarrow V_\gamma(\theta) = R$$

So $T_\gamma(\theta) = \dfrac{\dot{\gamma}(\theta)}{V_\gamma(\theta)} = -\sin\theta\cdot e_1 + \cos\theta\cdot e_2$

$$K_\gamma(\theta) = \frac{1}{V_\gamma(\theta)}\frac{d}{d\theta}T_\gamma(\theta) = -\frac{1}{R}\cdot(\cos\theta\cdot e_1 + \sin\theta\cdot e_2)$$

$$\kappa_\gamma(\theta) = \|K_\gamma(\theta)\| = \frac{1}{R}$$

Note: $\gamma(\theta) + r^2 K_\gamma(\theta) = p$ is an identity for a circle

(Using this relation: We will find for any curve the "best approximating" circle at every point.)

Remarks:

- For any curve γ (which C^2 and regular): At any point $\gamma(t)$, we will define:
  - Radius of curvature: $R = \frac{1}{k_\gamma(t)}$
  - Center of curvature: $p = \gamma(t) + R^2 \cdot K_\gamma(t)$

General formula relating the acceleration and the curvature:

Lem: If $\gamma: I \to \mathbb{R}^n$ is C^2 and regular, then

$$\ddot{\gamma}(t) = v_\gamma^2(t)\, K_\gamma(t) + \langle \ddot{\gamma}(t), T_\gamma \rangle\, T_\gamma$$

(In otherwords: $v_\gamma^2 \cdot \|K_\gamma\|$ is the perpendicular component of the acceleration)

Proof: $K_\gamma(t) = \frac{1}{v_\gamma} \frac{d}{dt}\left(\frac{\dot{\gamma}}{v_\gamma}\right) = \frac{\ddot{\gamma}}{v_\gamma^2} - \frac{\dot{\gamma}}{v_\gamma^3} \cdot \dot{v}_\gamma$ ①

Note: $\dot{v}_\gamma = \frac{d}{dt}\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} = \frac{\cancel{2}\langle \ddot{\gamma}, \dot{\gamma} \rangle}{\cancel{2}\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}} = \frac{\langle \ddot{\gamma}, \dot{\gamma} \rangle}{v_\gamma}$

So ① $\Rightarrow$ $K_\gamma = \frac{\ddot{\gamma}}{v_\gamma^2} - \frac{\dot{\gamma}}{v_\gamma^4} \langle \ddot{\gamma}, \dot{\gamma} \rangle$

The required relation follows noting that $T_\gamma = \frac{\dot{\gamma}}{v_\gamma}$ □